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ABSTRACT

This booklet includes short descriptions of the history of the calendar, Napier's Bones, and the beginnings of algebra. The remaining two stories discuss the number nine raised to the ninth power of nine, and repeating decimals. (DT)

5 LITTLE STORIES

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FIVE LITTLE STORIES

An Unbelievable Month of September

Napier's Bones

Why X Is Used for the Unknown

A Colossal, Enormous, Stupendous Number

The Strange Reciprocal of Seventeen

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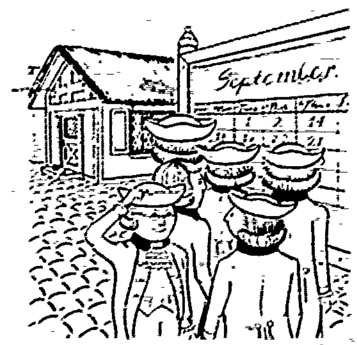
AN UNBELIEVABLE MONTH OF SEPTEMBER



Thirty days hath September, April, June, and November: all the rest have thirty-one. . . . That's fine for the present, but it was not always thus. At one time the year had only ten months. Witness the words *September, October, November* and *December*, which named the seventh month, the eighth month, the ninth month, and the tenth month. The calendar we follow today is the result of many changes. Now, we say "Merry Christmas" on December 25, and seven days later we say "Happy New Year." In 1750 the people of England and the American Colonies had to wait 90 days after Christmas to say "Happy New Year."

Trying to get a month based on occurrences of the moon to be compatible with a year based on occurrences of the sun gave calendar makers a great deal of trouble. Early peoples were familiar with one phase of the moon consisting of about 29½ days and one phase of the sun requiring about 365¼ days. Calendar makers could not say $M \times 29\frac{1}{2}$ days $= 365\frac{1}{4}$ days and have M a whole number. Once they did try 30 days in a month and 360 days in a year and could say $M \times 30$ days $= 360$ days and have M equal to the whole number 12. That seemed fine, but it didn't work. The people found that if the year consisted of 360 days, certain anniversary dates got more and more out of step with the seasons as time passed.

When Julius Caesar became Emperor of Rome, he found that the calendar then in use was far from satisfactory. With the aid of an Egyptian astronomer, Sosigenes, Caesar established in 45 B. C. what has since been called the Julian Calendar. This is much like our present one because common years had 365 days and leap years had 366 days. It was based on a year of exactly 365¼ days, that is, 365 days and 6 hours. Later, more careful calculations showed that a year consists of 365 days, 5 hours, 48 minutes, and 49.7 seconds, which is about 11 minutes less than what Sosigenes had calculated.

By 1582 the error between the calendar in use and the *true* calendar amounted to 10 days. During March of that year Pope Gregory XIII issued a brief in which he abolished the use of the Julian Calendar and substituted one which had been constructed by Aloysius Lilius (or Luigi Lilio Ghiraldi), a learned astronomer of Naples, Italy. This is our present calendar and it is called the Gregorian Calendar. Pope Gregory ordered the elimination of 10 days from the year 1582 by having the day following October 4 called October 15 instead of October 5.

It was not until 1752 that England and her Colonies changed from the Julian Calendar to the Gregorian Calendar. By that time an elimination of 11 days was required to make the change. An Act of Parliament in 1751

provided that in 1752, January 1 instead of March 25 should be taken as New Year's Day and that September 2 should be followed by September 14 instead of September 3.

Unless one has heard of these changes, the month of September 1752 is unbelievable. This month had only two Sundays and two Mondays. It contained 19 days instead of the customary 30 days. No one could write a letter or transact any business on September 10, for there was no such day. The first two days of the month were of the Julian Calendar (Old Style) and the last 17 days were of the Gregorian Calendar (New Style). A copy of the calendar for September 1752 is given at the right.

SEPTEMBER 1752

S	M	T	W	T	F	S
		1	2	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30

George Washington was born on February 11, 1732, Old Style. He celebrated his twenty-first birthday on February 22, 1753, New Style. He died December 14, 1799, New Style. It is somewhat puzzling to determine the number of years, months, and days that Washington lived. If anyone wishes to try it, remember to count the days in the leap years, Old Style and New Style, as well as the loss of 11 days in 1752.

Most of the years from now on will contain 365 days. Every year, however, whose number is exactly divisible by four will contain 366 days, except for the last year of each century; this is a leap year only when the *number of the century* is exactly divisible by four, except for the years 4000, 8000, and $n(4000)$, which will be regarded as common years of 365 days. Our present calendar is so nearly perfect that it will not require adjustments or changes for at least two hundred centuries.

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NAPIER'S BONES

If someone should ask you to name some of the outstanding men of Scotland, your list might contain the names of Sir William Wallace, John Knox, John Napier, James Watt, Robert Burns, Sir Walter Scott, Thomas Campbell, Thomas Carlyle, Sir James Barrie, and Dr. Alexander Fleming. Very possibly you might name some others, but I hope you would include John Napier, Laird of Merchiston. This story will tell briefly why his name should be respected and remembered, and it will describe and tell how to use Napier's Bones.

John Napier was born in Merchiston Castle on the outskirts of Edinburgh, Scotland, in 1550 and died there in 1617. The family name was variously spelled: Napier, Nepeir, Naper, Neper, Naperis, and Naperius. Hume, the Scotch historian and philosopher, wrote that Napier deserved the title "great man" more than any other born in Scotland. Napier planned and devised many inventions. He wrote a theological work called *A Plain Discovery of the Whole Revelation of Saint John,* which ran through several English editions. It was translated into French, Dutch, and German and was widely read for more than 40 years. He also won fame as an astronomer, as an engineer, as a physicist, and as a mathematician.

Napier was bold, courageous, and stubborn. He helped to advance the skills and knowledge of arithmetic, algebra, and plane and spherical trigonometry. The idea of logarithms belongs uniquely to Napier, and he worked more than 20 years to perfect his theory. A prominent mathematical historian stated that it came as a "bolt from the blue," for it was not connected in any way with any previous mathematical theory. Our present tables of common logarithms were developed by Henry Briggs after he had twice consulted with Napier. They agreed that a base of 10 for logarithms would prove more satisfactory in most cases than the base originally used by Napier.

The skills of arithmetic computation developed slowly. Early peoples used fingers and toes as aids. Soon pebbles and twigs or splints of wood were used. Marks made in sand spread on a floor or on a table, balls on strings or on wires to form counting frames, and special schemes of marking figures on squares such as are found on a chess board were some of the successive developments. Before 1500 A.D. a number of Italian mathematicians had made great strides in simplifying multiplication. One of the methods developed is illustrated in Figure 1. It is called the Gelosia Method of Multiplication, and it appeared in a book printed in the town of Treviso, Italy, in 1478. The word *gelosia* means grating or lattice.

The illustration shows the process used on the gelosia for multiplying 854 by 312. Observe that the product 3 × 4, or 12, appears in the upper right hand box with 1 (or the tens' digit) above the diagonal line and 2 (or the units' digit) below the diagonal line. Note, also, the product 2 × 8, or 16, which appears in the lower left hand box of the gelosia. The final product is found by adding diagonally downward: (8), (4 + 0 = 4), (2 + 5 + 1 + 6 = 14), (1 + 5 + 8 + 1 + [1] = 16), (1 + 4 + [1] = 6), and (2). Thus 854 × 312 = 266,448.

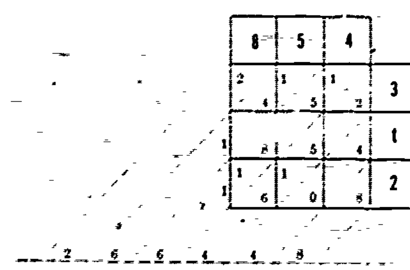

FIGURE 1

There seems to be no doubt that Napier was familiar with the Gelosia Method. He invented a system of rods arranged to utilize this method of multiplication. He explained how the rods were to be used in an article called *Rabdologia*, which means *A Collection of Rods*.

Napier made his rods of bone or ivory in the shape of square prisms, each about 3 inches long and about $\frac{3}{10}$ inch for each side of a cross-section. Figure 2 shows a representation of a complete set of Napier's Rods compactly assembled. One face of the rod on the right is divided into ten squares. The numbers which appear in these squares are used singly as the digits of the multiplier. All four faces of the other ten rods are also divided into squares which are subdivided by diagonals, except the top squares. The numbers in the top squares are used as the digits of the multiplicand. Figures 3 and 4 picture the upper faces of two of the rods. Figure 5 shows how one of the rods appears in perspective.

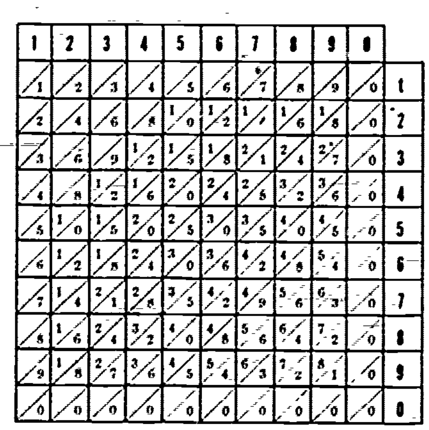

FIGURE 2

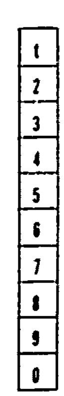

FIGURE 3

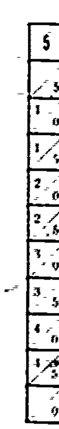

FIGURE 4

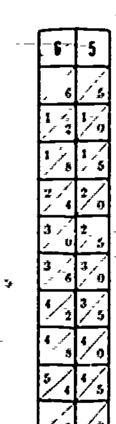

FIGURE 5

Napier's Rods are often called Napier's Bones because of the material from which they are made and because of the title of a book published in London in 1667 by W. Leybourn. He used the title *The Art of Numbring*

by *Speaking-Rods; Vulgarly Termed Napeir's Bones.* In the United States, the last part of the title would read *Commonly Called Napier's Bones.*

During the seventeenth century Napier's Bones attracted wide attention throughout Europe. They were also copied and used in China and Japan. Today they are of little or no practical value, although they remain as a curiosity of the past. More important, they are an example of man's continual attempt to reduce the drudgery of mathematical computations.

In order to appreciate the facility and utility of multiplying by using Napier's Bones, let us try a brief example. Following the scheme that Napier used, let us multiply 3742 by 68. Select the four rods with top numbers of 3, 7, 4, and 2. Now arrange the rods in the order of 3-7-4-2, reading from the left. Next place the rod containing 1, 2, 3, · · · , 8, 9, 0 upright along the right side of the rods already assembled, as shown in Figure 6.

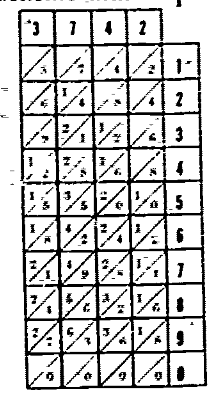

FIGURE 6

Place a ruler or the straight edge of a card, on Figure 6, below the line of figures ending with 8. This line when added diagonally downward, as shown in Figure 7, gives the product of 3742 and 8. Write this product, which is 29,936, on a piece of paper. Return to Figure 6. Place the ruler or card below the line of figures ending with 6. This line when added diagonally downward, as shown in Figure 8, gives the product of 3742 and 6. This product, which is 22,452, should be written on the piece of paper already used, under 29,936 but shifted over one place to the left, as shown in Figure 9. Then these partial products are added to find the product of 3742 and 68, which is 254,456.

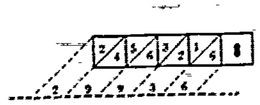

FIGURE 7

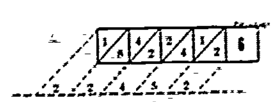

FIGURE 8

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   2 9 9 3 6
 2 2 4 5 2
 2 5 4 4 5 6
```

FIGURE 9

Please observe that the product of a four-digit number (3742) and a two-digit number (68) was found by combining two partial products and by the careful handling of Napier's Bones.

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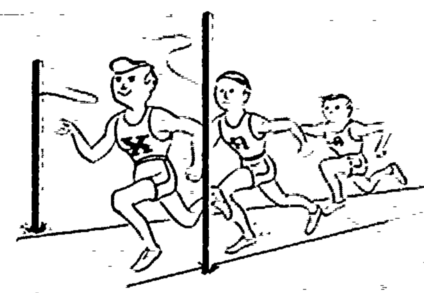
WHY X IS USED FOR THE UNKNOWN



Mathematicians may use a variety of letters to stand for the unknown. The letters b, c, m, x, z, and others are in use. However, the letters at or near the end of the alphabet are preferred, and x is used more often than y or z. Sometimes, θ (theta), ρ (rho), z (zed), $\aleph$ (aleph), and others are used in some fields of mathematics to stand for certain general and unknown numbers. Descartes, when he extended the familiar idea of latitude and longitude to determine the position of a point in a plane, called the coordinates of the point x and y. This is a different use of x from that found in the statement, "Let x equal the number of boxes of candy that Mary sold." This article deals with the x which is found in first year algebra and not with the x which is found in coordinate geometry.

Many persons like to use n or N to stand for the unknown in solving verbal problems and their equations, and in writing simple formulas. Of these two letters, n is used more often than N, because N may be used to stand for such words as North, Noon, and Nitrogen. Some prefer n to x because every time a problem is solved a number is sought, and n is the initial letter of the word *number*. The use of n might help one to keep in mind that he is dealing with numbers. Here are two examples of what the use of n will help to correct. Pupils have been heard to say, "Let x equal the strawberries that Sandra picked" and "x stands for the newspapers that Rodman sold." Of course pupils do not get strawberries or newspapers when they solve problems. To some pupils, x seems very abstract and mechanical. There are teachers who believe that the defining statements of a problem may be made more particular and related in the minds of pupils, especially first year algebra pupils, if n instead of x is used to stand for the unknown number.

It is generally agreed that x, or n, or any other letter or symbol does not necessarily stand for an unknown number. Sometimes the number sought is already known. Take this simple problem: "John has half as much money as Harry. Together they have 15 cents. How much has each?" We know immediately that John has 5 cents and that Harry has 10 cents. No matter what letter is used to represent the number of cents that John has, that number is not unknown. Such problems are not taught for the most part to find answers; that is, to find numbers. Such problems are taught as easy exercises in thinking—thinking to find relationships, thinking to choose proper words, thinking to compose concise and precise statements. Problems are taught as exercises in thinking.

Although some may not approve of using x and saying "the unknown," such usages do find wide acceptance; x is said to stand for the unknown

because this expression is a relic of the past. The term *relic* is used to indicate that the expression is something to be "venerated by the faithful." The development of the use of x for the unknown has a long and noble history which includes bits of the lives and works of outstanding mathematicians.

About 1600 B.C. an Egyptian scribe and priest called Ahmes gathered together all the mathematics he could find and wrote a book on a scroll of crushed reeds called papyrus. He used the title *Knowledge of All Dark Things*. Many persons, especially high school pupils, might think that this is an excellent title for his book. Ahmes wrote principally about arithmetic, although some parts of his work dealt with algebra. One problem he discussed was: "A heap and its third equals 8. Find the heap." Today we would say, "If $x + x/3 = 8$, find x." Some translators say that Ahmes used "mass" and not "heap" in discussing his problem. Either word, however, implies that a definite, finite, measurable quantity is sought which has not as yet been defined or stated. Here began the use of a symbol—a word is a symbol—to stand for the implicit number of a problem.

About 800 A.D. another book was written which holds an important place in the history of mathematics. It was written by al-Khowarizmi and bore the title *'ilm al-jebr wa'l muqabalah*. The book dealt with restoring roots in equations and with simplifying mathematical expressions. Observe the word *al-jebr* in the title of the book. It is from this word that the English word *algebra* is derived. Al-Khowarizmi used the symbol *root* to stand for the implicit number, that is, the unknown in equations.

Early Latin writers continued to use the word *radix* (root) for the number sought in equations and problems. Sometime later, *res* (thing) was used by Latin writers. The words *mass, heap,* and *thing* have two common qualities. Each is a symbol. Each stands for a measurable quantity whose exact value is not stated. Today in ordinary language we use the words *thing* and *something* in that same sense. Early Italian writers used *cosa*. Early German writers used *coss*. The Arabian writers used *šai* (pronounced *shei* or *shay*). The words *res, cosa, coss,* and *šai* are translated into English as *thing*.

From about 1000 A.D. to 1400 A.D. classical learning came into Europe with the Moors (Saracens) from Africa by way of Spain. Christian scholars in the Spanish monastaries reconstructed and translated Greek, Latin, and Arabic manuscripts. The so-called unknown quantity of algebra, which was written in the Arabic manuscripts as *šai*, was translated by the Spanish writers as *χei* (thing). The initial of this word resembled the Greek letter, χ (Chi). In early Spanish the character χ was pronounced as *sh*. Therefore, the Arabic *šai* and the early Spanish *χei* were pronounced alike and each word meant *thing*. Early English writers accepted the word *χei* as it came to them in the Spanish manuscripts to stand for the unknown. Later they used the initial χ of the word for that purpose. Eventually the English writers used x because of the resemblance of this letter to the Spanish χ. So x came to represent the number sought in the solution of equations and problems. It would seem then that mathematicians and student of mathematics in using x for the unknown have been respecting an honored tradition and following an old Spanish custom.

To conclude this story, two authoritative references are given to substantiate the account of the origin of the use of x to stand for the unknown.

1. WEBSTER'S NEW INTERNATIONAL DICTIONARY, Second Edition, Unabridged, 1953, page 2962:

> X, x . . . An unknown quantity. *X* was used as an abbreviation for Ar. [Arabic] *shei* a thing, something, which, in the Middle Ages, was used to designate the unknown, and was then prevailingly transcribed as *xei*.

2. LOKOTSCH, ETYMOLOGISCHES WOERTERBUCH EUROPAEISCHER WÖRTER ORIENTALISCHEN URSPRINGS, ART. 1770:

> Arab. šai., Ding, Sache. So bezeichneten die arabischen Mathematiker die unbekannte und verwandten die Abkuerzung 'š', die von den Spaniern durch 'x' wiedergegeben wurde, da dieser Buckstäbe im aelteren Spanischen so ausgesprochen wurde. Daher wird die unbekannte Groesse in der Algebra mit 'x' bezeichnet.

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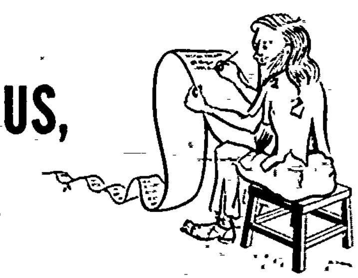
A COLOSSAL, ENORMOUS, STUPENDOUS NUMBER



Upon reading the title, some may think that this little story will talk about infinity (∞). Infinity is not a number as we ordinarily or extraordinarily understand numbers. The natural numbers are used to count one, two, three, · · · , one million, · · · , one octillion, · · · ; one never comes to an end and says infinity, for there is no end to counting. No matter with what number one may wish to stop, there is always one more which can be added. So infinity does not terminate the sequence of natural numbers used in counting.

Mathematics books discuss other numbers which are extraordinary, to say the least; for example, $\sqrt{2} = 1.414$,* $\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$, $\pi = 3.142$, $i = \sqrt{-1}$, $e = 2.718$, and $M = .434$ are not natural numbers. They are numbers, but none of them is infinity. Infinity is not a number at all. It's simply a short way of saying "great beyond all measure." And so this story is not about infinity.

What is this colossal, enormous, stupendous number? It is a definite, finite number. It is exactly so much and no more, but it is very great. Not great beyond all measure, but great beyond normal comprehension. The number to be discussed is 9 raised to the power of the ninth power of 9. Symbolically this number is written $9^{(9^9)}$

Before trying to describe the size of this number it might be advisable to review briefly some elementary properties of exponents. A number placed slightly above and to the right of another number is called its exponent. The exponent tells how many times the number is to be taken as a factor. For example, $2^3 = 2 \times 2 \times 2$, $8^2 = 8 \times 8$, $7^4 = 7 \times 7 \times 7 \times 7$, $5^9 = 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5$.

The expression *raised to the power of* is often used in connection with exponents: 2 is raised to the power of 3 in 2^3, 5 is raised to the power of 9 in 5^9. Sometimes, 2^3 and 5^9 are more simply stated as the third power of 2 and the ninth power of 5. The numbers $(3^3)^3$ and $3^{(3^3)}$ look somewhat alike and the expressions used to read them sound somewhat alike. The first may be read three to the third power raised to the third power. The second may be read three raised to the third power of three. But these numbers are by no means equal in value.

The first of these numbers, $(3^3)^3$, means $(3 \times 3 \times 3)^3$ or $(3 \times 3 \times 3) \times (3 \times 3 \times 3) \times (3 \times 3 \times 3)$ or the ninth power of 3. The second of the

* These values are correct to three decimal places.

numbers, $3^{(3^3)}$, means 3^{27} or the twenty-seventh power of 3. If the expanded values of these numbers are found by actual multiplication and the value of the second is divided by the value of the first, this will show that $3^{(3^3)}$ is 387,420,489 times as large as $(3^3)^3$. One might think that by like calculations it would be possible to determine the relative sizes of $(9^9)^9$ and $9^{(9^9)}$. This is not true, for it cannot be done within a lifetime by performing the actual operations of multiplication and division. However it can be determined by other means that $(9^9)^9$ is relatively a small number when compared with $9^{(9^9)}$. Our story concerns the number $9^{(9^9)}$ which is colossal, enormous, and stupendous. We shall examine its size but never find its expanded value, for it is far too large.

When one raises a number to a given power, the number of multiplications needed is one less than the value of the exponent. For example, $5^2 = 5 \times 5$ requires only one multiplication; $6^4 = 6 \times 6 \times 6 \times 6$ requires only three multiplications. So, $9^{(9^9)}$ or $9^{387,420,489}$ requires 387,420,488 multiplications. If a person can average two multiplications a minute for 24 hours a day without pause, 365 days a year, it would take more than 368½ years to complete the multiplications. This is one way of showing how large the number $9^{(9^9)}$ is. Its size in expanded form is beyond normal comprehension.

We are interested not in which figures, but in how many figures there are in the expanded result. We have found that we cannot learn how many figures there are by repeated multiplications because we do not have the time. There is a method, however, of finding the length of the expanded result without finding the result itself. This method will use some elementary properties of logarithms and log 9 = 0.95424 25094. (The value of log 9 correct to 20 decimal places is 0.95424 25094 39324 87459.)

Even though you may not have studied logarithms, you should be able to comprehend the calculations which follow. Previously we found that $9^{(9^9)} = 9^{387,420,489}$. By expressing the logarithm of each side, we have $\log 9^{(9^9)} = \log 9^{387,420,489}$. In the study of logarithms we learn that "the logarithm of a number raised to any power is equal to the logarithm of the number multiplied by the index of the power." Therefore, $\log 9^{387,420,489} = 387,420,489 \times \log 9$. If we substitute the value of log 9 in the right hand member of this equation, we may write $\log 9^{387,420,489} = 387,420,489 \times 0.95424\ 25094$. By actual multiplication, $387,420,489 \times 0.95424\ 25094 = 369,693,099.6 \cdots$ Finally, then, $\log 9^{(9^9)} = 369,693,099.6 \cdots$. Only the whole number to the left of the decimal point in the right hand member of this equation is important to the work at hand. This number plus one tells us how many digits there are in our final answer. In other words, there are 369,693,099 + 1 or 369,693,100 digits in the decimal expansion of $9^{(9^9)}$.

The 1942 printing of the *Encyclopedia Britannica* required 24 volumes. There are approximately 1000 pages to each volume. Each page contains two columns of 72 lines each. In each line in either column, there are about 65 letter spaces or figure spaces. By multiplying $1000 \times 72 \times 2 \times 65$,

it is found that one volume of the *Encyclopedia* contains space for approximately 9,360,000 figures. To print the answer of $9^{(9^9)}$, which requires 369,693,100 figures, more than 39 such volumes would be required. The tremendous magnitude of this number may, possibly, be better appreciated if we say that to print the answer of $9^{(9^9)}$ one complete set of the *Encyclopedia* and more than 15 additional volumes more are required. This number is bigger than big.

It has been stated that a French scientist calculated that a sphere of platinum one million light years in radius would contain atoms to the number of 225 followed by 88 ciphers; that is, a number of 91 digits. $9^{(9^9)}$ is a number which requires over four million times as many digits. It is believed that $9^{(9^9)}$ is so large that it is sufficient to number all the atoms of creation. *It is simply out of this world.*

There are larger numbers, of course. It is the largest, however, which can be represented by three digits. But far more important than that, $9^{(9^9)}$ is an outstanding example of the simplicity, the compactness, and the power of the symbolism of mathematics.

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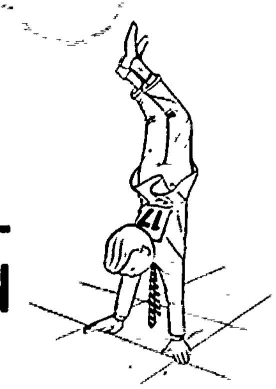
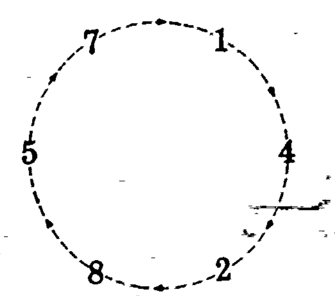
THE STRANGE RECIPROCAL OF SEVENTEEN



The reciprocal of a number is 1 divided by that number. For example, $1 \div 2$ or $\frac{1}{2}$ is the reciprocal of 2, and $\frac{1}{17}$ is the reciprocal of 17. The reciprocals of the first ten positive whole numbers expressed as fractions and as decimals to six places are: $\frac{1}{1} = 1.000{,}000$; $\frac{1}{2} = .500{,}000$; $\frac{1}{3} = .333{,}333\frac{1}{3}$; $\frac{1}{4} = .250{,}000$; $\frac{1}{5} = .200{,}000$; $\frac{1}{6} = .166{,}666\frac{2}{3}$; $\frac{1}{7} = .142{,}857\frac{1}{7}$; $\frac{1}{8} = .125{,}000$; $\frac{1}{9} = .111{,}111\frac{1}{9}$; $\frac{1}{10} = .100{,}000$. These values are of three kinds. In the expression of the decimal values of $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{8}$, and $\frac{1}{10}$, the zero is repeated several times at the right of the decimal point. Any number of these zeros could be omitted and still the decimal value would be complete. A second kind takes in the values $\frac{1}{3}$, $\frac{1}{6}$, and $\frac{1}{9}$ which repeat a single figure, not zero, a number of times and require a fraction to complete their values. The value of the reciprocal of seven, the fraction $\frac{1}{7}$, is the third kind. It shows six different figures to the right of the decimal point and needs a final $\frac{1}{7}$ to complete its value. This final $\frac{1}{7}$ means that the same group of six figures and the final $\frac{1}{7}$ will be repeated and annexed to the first group of six figures if the number of decimal places is extended and the value is to be accurate. For example, $\frac{1}{7} = .142857\frac{1}{7} = .142857142857\frac{1}{7}$. The value of $\frac{1}{7}$ might be written also as $.142857142857142857 \cdots 142857142857\frac{1}{7}$. The decimal value of $\frac{1}{7}$ is called a repeating or a circulating decimal with a repeating period of six figures. Of course, the values for $\frac{1}{5}$ and $\frac{1}{6}$ are repeating decimals also. In $\frac{1}{5} = .200{,}000 \cdots$ the single figure 0 is repeated and in $\frac{1}{6} = .166{,}666 \cdots$ the single figure 6 is repeated. In $\frac{1}{7} = .142857142857 \cdots$ the entire group of six figures is repeated. The figure or the group of figures which is repeated over and over again is called a *repetend*. When there is a single figure as a repetend, it is simply indicated by placing a dot over that figure; for example, $\frac{1}{3} = .\dot{3}$ and $\frac{1}{6} = .1\dot{6}$. When there is more than one figure in the repetend, it is indicated by placing a dot over the initial figure and another dot over the terminal figure of the repetend; for example, $\frac{1}{7} = .\dot{1}4285\dot{7}$ and $\frac{1}{17} = .\dot{0}58823529411764\dot{7}$.

If the decimal value of the reciprocal of 7 ($.142857\frac{1}{7}$) is multiplied by 2, 3, 4, 5, and 6, some interesting numbers are produced. The products are equal respectively to $.285714\frac{2}{7} = .\dot{2}8571\dot{4}$, $.428571\frac{3}{7} = .\dot{4}2857\dot{1}$, $.571428\frac{4}{7} = .\dot{5}7142\dot{8}$, $.714285\frac{5}{7} = .\dot{7}1428\dot{5}$, $.857142\frac{6}{7} = .\dot{8}5714\dot{2}$. If the five repetends $.\dot{2}8571\dot{4}$, $\dot{4}2857\dot{1}$, $.\dot{5}7142\dot{8}$, $.\dot{7}1428\dot{5}$, and $.\dot{8}5714\dot{2}$ are examined, it will be found that these numbers use the very same digits which are found in the decimal value of $\frac{1}{7}$ or $.\dot{1}4285\dot{7}$. More than that, the order in which these digits are arranged follows a definite and fixed scheme.

The accompanying circle, with the digits of the decimal value of the reciprocal of seven arranged in clockwise order around the circle, will assist in showing what this scheme is. In writing the product of $5 \times .\dot{1}4285\dot{7}$, one can understand quite easily that the answer ends in 5. Observing the circle, it is not difficult to see that the product begins with 7. The answer is read in clockwise order from the circle as $7 \rightarrow 1 \rightarrow 4 \rightarrow 2 \rightarrow 8 \rightarrow 5$; that is, $5 \times 1/7 = .\dot{7}1428\dot{5}$. In finding the result of multiplying $.\dot{1}4285\dot{7}$ by 6, it is evident that the product ends in 2 and begins with 8.



Reading from the number itself or from the circle, the answer is found as $8 \rightarrow 5 \rightarrow 7 \rightarrow 1 \rightarrow 4 \rightarrow 2$ or $.\dot{8}5714\dot{2}$. The decimal value of the reciprocal of 7 is said to be a repeating decimal of six figures arranged in cyclic order.

There are not many of these strange cyclic reciprocals. The next one in line is the reciprocal of seventeen or $1/17 = .\dot{0}58823529411764\dot{7}$. This is a repeating decimal of 16 figures arranged in cyclic order. Having become familiar with the decimal value of the reciprocal of 7 and its properties, we can easily find that $2 \times 1/17$ or $2 \times .\dot{0}58823529411764\dot{7}$ equals either $.705882352941176\dot{4}$ or $.\dot{1}17647058823529\dot{4}$, since the result must end in 4 and since there are two 4's in repetend for $1/17$. The second of these possibilities is chosen because the first three figures of the repetend, .058, when multiplied by 2 equals .116. The product sought could not begin with .7. Possibly the reader would like to try to write the answer of $9 \times 1/17$ or $9 \times .\dot{0}58823529411764\dot{7}$ without performing the actual multiplication. Here the answer must end with the figure 3. It begins with $.529 \cdots$. If you try any other multiplications with the reciprocal of 17, your multiplier should not be greater than 16, *although it is possible to use greater multipliers and still use the strange properties of the reciprocal of 17.*

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